ELSEVIER
Discrete Mathematics 131 (1994) 287–299
DISCRETE MATHEMATICS

Signs in the Laplace expansions and the parity of the distinguished representatives

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Received 23 January 1992; revised 11 March 1992

Abstract

In this paper, the signs of the terms in the Laplace expansions of determinants are investigated. The problem is equivalent to that of the parity of the distinguished coset representatives for the (maximal) parabolic subgroups of the symmetric groups. The number of positive and negative terms (equivalently, the number of even and odd representatives) are given explicitly together with their generating functions. Also mentioned are some applications to the symbolic method (in invariant theory) and combinatorics.

1. Introduction

The Laplace expansion theorem, a (to some extent, the) fundamental theorem of determinants and multilinear algebra, states that an $n\times n$ determinant can be expressed as a sum of $\binom{n}{k}$ terms, each of which being the product of a $k\times k$ minor and an $(n-k)\times(n-k)$ minor with a sign (cf. [4, 5])

$$\det\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn}\end{pmatrix} = \sum_{1\leqslant j_1<\cdots<j_k\leqslant n} (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k} \det\begin{pmatrix} x_{i_1j_1} & \cdots & x_{i_1j_k} \\ \vdots & & \vdots \\ x_{i_kj_1} & \cdots & x_{i_kj_k}\end{pmatrix}$$

$$\times\det\begin{pmatrix} x_{i'_1j'_1} & \cdots & x_{i'_1j'_{n-k}} \\ \vdots & & \vdots \\ x_{i'_{n-k}j'_1} & \cdots & x_{i'_{n-k}j'_{n-k}}\end{pmatrix}, \tag{1}$$

where $1\leqslant i_1<\cdots<i_k\leqslant n$ and $1\leqslant i'_1<\cdots<i'_{n-k}\leqslant n$ with $\{i_1,\cdots,i_k\}\cup\{i'_1,\cdots,i'_{n-k}\}=\{1,\cdots,n\}$ are chosen arbitrarily but fixed, the sum is over all possible arrangements $1\leqslant j_1<\cdots<j_k\leqslant n$ with $\{j_1,\cdots,j_k\}\cup\{j'_1,\cdots,j'_{n-k}\}=\{1,\cdots,n\}$ and $1\leqslant j'_1<\cdots<j'_{n-k}\leqslant n$. If the sign is $+$ (resp. $-$) in front of a term in (1), i.e., if $i_1+\cdots+i_k+j_1+\cdots+j_k$ is even (resp. odd), we say that this term is *positive* (resp.

SSDI 0012-365X(92)00548-M

negative). It is interesting to know how many terms are positive, how many negative, and if we can have (say) one million more positive terms than negative terms, etc. The answers will be given in Section 3. In Section 2 we give a description of the distinguished representatives and their role in the Laplace expansion. A very short proof of the Laplace expansion theorem is also included. Some generating functions are deduced in Section 4 and several applications are given in Section 5.

2. The distinguished representatives

For each positive integer n, let S_n be the symmetric group on n letters $1, 2, \ldots, n$. For k, $0 \leqslant k \leqslant n$, $S_k \times S_{n-k}$, denoted by $H_{n,k}$ (or simply H_k if no confusion), is canonically imbedded in S_n (via separately permuting the first k and the last $n-k$ letters). For $\sigma \in S_n$, $i(\sigma)$ is the number of inversions in σ. It is not hard to show that $i(\sigma)$ is the minimal number of transpositions $(1, 2), (2, 3), \ldots, (n-1, n)$ needed (allowing repetitions) to express σ as their product.

Lemma 2.1. *Each H_k has a set of minimal left-coset representatives $\sigma_1, \ldots, \sigma_{\binom{n}{k}}$ in the following sense. Every $\sigma \in S_n$ can be written uniquely as $\sigma = \sigma_j h$ with $h \in H_k$, $i \leqslant j \leqslant \binom{n}{k}$, and $i(\sigma) = i(h) + i(\sigma_j)$.*

Proof. Given σ, take σ_j such that $\{\sigma_j(1), \ldots, \sigma_j(k)\} = \{\sigma(1), \ldots, \sigma(k)\}$ and $\sigma_j(1) < \cdots < \sigma_j(k)$, $\sigma_j(k+1) < \cdots < \sigma_j(n)$. Then h is also determined $(= \sigma_j^{-1}\sigma \in H_k)$. From this, the rest is clear. $\square$

$\sigma_1, \ldots, \sigma_{\binom{n}{k}}$ are called the *distinguished* (*left-coset*) *representatives.*

Example

$$n=5, \quad k=2, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix} h,$$

where $h = (1, 2)(3, 5, 4) \in H_2$.

Remark. In the Lie-theory language, H_k is a "parabolic subgroup" (a subgroup generated, up to conjugation, by a subset of the "simple reflections" $(1, 2)$, $(2, 3), \ldots, (n-1, n)$) of the "coxeter group" S_n. We also have the distinguished representatives for parabolic subgroups in a coxeter group, of which the above lemma is special case. See [1, Ch. IV, Section 1, Exercises 3 and 4].

For the sake of simplicity (of notation), we assume that we are expanding the determinant along the *first* k rows (i.e., $i_1 = 1, \ldots, i_k = k$.); see Proposition 3.9 below for the general case.

Following the notation in [2], $[s_1 \ldots s_m \mid t_1 \ldots t_m]$ denotes

$$\det\begin{pmatrix} x_{s_1t_1} & \cdots & x_{s_1t_m} \\ \vdots & & \vdots \\ x_{s_mt_1} & \cdots & x_{s_mt_m} \end{pmatrix}.$$

Now we can rewrite (1) in terms of the σ_j's.

Proposition 2.2. *Let $\sigma_1, \ldots, \sigma_{\binom{n}{k}}$ be the distinguished representatives. Then*

$$[1\ 2 \cdots n \mid 1\ 2 \cdots n] = \sum_{j=1}^{\binom{n}{k}} (-1)^{i(\sigma_j)} [1 \cdots k \mid \sigma_j(1) \cdots \sigma_j(k)] \times [k+1 \cdots n \mid \sigma_j(k+1) \cdots \sigma_j(n)]. \tag{2}$$

Thanks to our set-up, proof of (2) will be very concise, which we cannot resist presenting.

Proof of Proposition 2.2. For any $\tau \in S_n$, let $\tau(x_{i_1j_1}x_{i_2j_2} \cdots)$ denote $x_{i_1\tau(j_1)}x_{i_2\tau(j_2)} \cdots$. So by definition of determinant,

$$[12 \cdots n \mid 12 \cdots n] = \sum_{\sigma \in S_n} (-1)^{i(\sigma)} \sigma(x_{11}x_{22} \cdots x_{nn}).$$

By the coset decomposition

$$S_n = \bigcup_{j=1}^{n} \sigma_j(S_k \times S_{n-k}), \qquad \sum_{\sigma \in S_n} = \sum_{j=1}^{\binom{n}{k}} \sum_{\sigma' \in S_k} \sum_{\sigma'' \in S_{n-k}}.$$

Also, $(-1)^{i(\sigma_j\sigma'\sigma'')} = (-1)^{i(\sigma_j)}(-1)^{i(\sigma')}(-1)^{i(\sigma'')}$. $\sum_{\sigma' \in S_k} (-1)^{i(\sigma')}\sigma'$ and $\sum_{\sigma'' \in S_{n-k}} (-1)^{i(\sigma'')}\sigma''$ contribute to $[1 \cdots k \mid 1 \cdots k]$ and $[k+1 \cdots n \mid k+1 \cdots n]$, respectively, by the definition of determinant. So

$$[1 \cdots n \mid 1 \cdots n] = \sum_{j=1}^{\binom{n}{k}} (-1)^{i(\sigma_j)} \sigma_j [1 \cdots k \mid 1 \cdots k][k+1 \cdots n \mid k+1 \cdots n]$$

$$= \sum_{j=1}^{\binom{n}{k}} (-1)^{i(\sigma_j)} \sigma_j [1 \cdots k \mid \sigma_j(1) \cdots \sigma_j(k)][k+1 \cdots n \mid \sigma_j(k+1) \cdots \sigma_j(n)]. \quad \square$$

Let $P_{n,k}$ (resp. $N_{n,k}$) be the number of even (resp. odd) σ_j's, $1 \leqslant j \leqslant \binom{n}{k}$. So in the Laplace expansion (2), there are $P_{n,k}$ positive terms and $N_{n,k}$ negative terms.

Remark. It is easy to see that (2) is true for *arbitrary* coset representatives. We use the distinguished ones for the certainty and naturalness of the expression as well as the clarity of argument (e.g., see proof of Proposition 3.3).

In the next two sections we will study $P_{n,k}, N_{n,k}$ and their difference $D_{n,k} = P_{n,k} - N_{n,k}$.

3. $P_{n,k}$, $N_{n,k}$ and $D_{n,k}$.

We start with some obvious observations.

Proposition 3.1. *The numbers* $P_{n,k}$ *and* $N_{n,k}$ *satisfy*

(a) $P_{n,k} \geqslant 0$, $N_{n,k} \geqslant 0$.

(b) $P_{n,k} + N_{n,k} = \binom{n}{k}$, $\quad \sum_{k=0}^{n} P_{n,k} + \sum_{k=0}^{n} N_{n,k} = 2^n$.

(c) $P_{n,n-k} = P_{n,k}$, $\quad N_{n,n-k} = N_{n,k}$, $\quad$ *if* $k(n-k)$ *is even.*
$P_{n,n-k} = N_{n,k}$, $\quad N_{n,n-k} = P_{n,k}$, $\quad$ *if* $k(n-k)$ *is odd.*

Proof. (a) and (b) are obvious. For (c), let

$$\hat{\sigma} = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ k+1 & k+2 & \cdots & n & 1 & \cdots & k \end{pmatrix},$$

and it is easy to check that $\sigma_1 \hat{\sigma}, \ldots, \sigma_{\binom{n}{k}} \hat{\sigma}$ are the distinguished representatives for H_{n-k}, and that $\hat{\sigma}$ is odd if and only if $k(n-k)$ is odd. $\square$

Remark. Later we will see (Proposition 3.5) that $P_{n,n-k} = P_{n,k}$ and $N_{n,n-k} = N_{n,k}$ for *all* n and k.

The following "boundary values" are also trivial.

Proposition 3.2. *For any* n, *we have that* $P_{n,0} = P_{n,n} = 1$, $N_{n,0} = N_{n,n} = 0$, *and* $D_{n,0} = D_{n,n} = 1$.

Now we deduce some recurrence relations for $P_{n,k}$, $N_{n,k}$ and $D_{n,k}$, which will enable us to get closed formulas later.

Proposition 3.3. *If* k *is even, we have*

(a) $P_{n,k} = P_{n-1,k-1} + P_{n-1,k}$,

(b) $N_{n,k} = N_{n-1,k-1} + N_{n-1,k}$,

(c) $D_{n,k} = D_{n-1,k-1} + D_{n-1,k}$.

If k *is odd, we have*

(a)′ $P_{n,k} = P_{n-1,k-1} + N_{n-1,k}$,

(b)′ $N_{n,k} = N_{n-1,k-1} + P_{n-1,k}$,

(c)′ $D_{n,k} = D_{n-1,k-1} - D_{n-1,k}$.

Proof. Each distinguished representative must be of the form

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ 1 & & \cdots & & & \cdots & \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ & & \cdots & & 1 & \cdots & \end{pmatrix}.$$

There are $P_{n-1,k-1}$ of the former type, and $P_{n-1,k}$ or $N_{n-1,k}$ of the latter type depending on k is even or odd. □

Remark. By adding (a) and (b), or adding (a)′ and (b)′, we get the familiar $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$.

By repeating Proposition 3.3, we get the following result.

Proposition 3.3′. *If k is even, we have*

(a) $P_{n,k}=P_{n-1,k-1}+P_{n-2,k-1}+\cdots+P_{k-1,k-1}$,
(b) $N_{n,k}=N_{n-1,k-1}+N_{n-2,k-1}+\cdots+N_{k-1,k-1}$,
(c) $D_{n,k}=D_{n-1,k-1}+D_{n-2,k-1}+\cdots+D_{k-1,k-1}$.

If k is odd, we have

(a)′ $P_{n,k}=P_{n-1,k-1}+N_{n-2,k-1}+P_{n-3,k-1}+\cdots$,
(b)′ $N_{n,k}=N_{n-1,k-1}+P_{n-2,k-1}+N_{n-3,k-1}+\cdots$,
(c)′ $D_{n,k}=D_{n-1,k-1}-D_{n-2,k-1}+D_{n-3,k-1}-\cdots$.

Using Propositions 3.2 and 3.3(c), we can get more relations among $D_{n,k}$.

Proposition 3.4. *For integers $l\geqslant m\geqslant 0$,*
$(\mathrm{A})_m$: *If $k=2m$ is even, then $D_{2l,k}=D_{2l+1,k}$.*
$(\mathrm{B})_m$: *If $k=2m+1$ is odd, then $D_{2l,k}=0$.*

Proof. The following induction will do:

Proposition 3.2 $\Rightarrow(\mathrm{A})_0\Rightarrow(\mathrm{B})_0\Rightarrow(\mathrm{A})_1\Rightarrow(\mathrm{B})_1\Rightarrow\cdots$. □

Proposition 3.4$(\mathrm{B})_m$ enables us to get rid of the unpleasant form in Proposition 3.1(c):

Proposition 3.5. *If n is even and k is odd, then*
(a) $D_{n,k}=0$,
(b) $P_{n,k}=N_{n,k}$.

Therefore, for all n and k, $P_{n,n-k}=P_{n,k}$, $N_{n,n-k}=N_{n,k}$ and $D_{n,n-k}=D_{n,k}$.

Using Propositions 3.2 and 3.3, we can arrange $P_{n,k}$'s, $N_{n,k}$'s and $D_{n,k}$'s in triangles (á la Pascal) as shown in Figs. 1, 2 and 3, respectively.

The triangle for $D_{n,k}$ is the more interesting one (somewhat fascinating).
Here the $2m$th row is the mth row of the Pascal triangle with 0 added in every other place; the $(2m+1)$st row is the mth row of the Pascal triangle with each entry repeated twice! What is more amazing (and amusing) is that if we draw "double-spaced" diagonals, we get the diamond shown in Fig. 4, which is obtained from the Pascal

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 1 \\
1 \quad 2 \quad 4 \quad 2 \quad 1 \\
1 \quad 3 \quad 6 \quad 6 \quad 3 \quad 1 \\
1 \quad 3 \quad 9 \quad 10 \quad 9 \quad 3 \quad 1 \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot
\end{array}
$$

Fig. 1. $P_{n,k}$

$$
\begin{array}{c}
0 \\
0 \quad 0 \\
0 \quad 1 \quad 0 \\
0 \quad 1 \quad 1 \quad 0 \\
0 \quad 2 \quad 2 \quad 2 \quad 0 \\
0 \quad 2 \quad 4 \quad 4 \quad 2 \quad 0 \\
0 \quad 3 \quad 6 \quad 10 \quad 6 \quad 3 \quad 0 \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot
\end{array}
$$

Fig. 2. $N_{n,k}$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 0 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 0 \quad 2 \quad 0 \quad 1 \\
1 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1 \\
1 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 1 \\
1 \quad 1 \quad 3 \quad 3 \quad 3 \quad 3 \quad 1 \quad 1 \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot
\end{array}
$$

Fig. 3. $D_{n,k}$

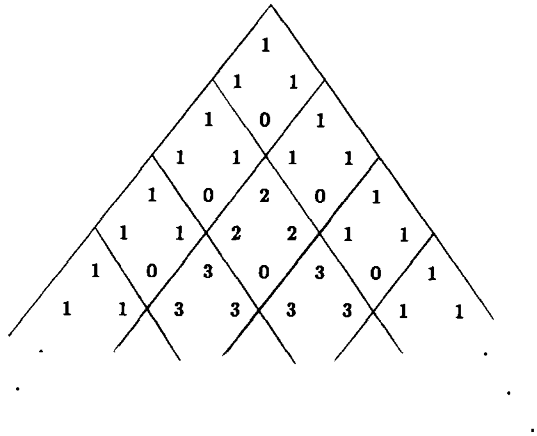

Fig. 4. The "diamonds"

triangle by replacing each "entry" $\binom{n}{k}$ by a "diamond"

$$\begin{array}{ccc} & \binom{n}{k} & \\ \binom{n}{k} & & \binom{n}{k} \\ & 0 & \end{array}$$

From this triangle (i.e., by applying Propositions 3.2–3.5), we get the following properties of $D_{n,k}$ that are not that obvious.

Proposition 3.6. (a) *All* $D_{n,k} \geqslant 0$, *i.e.*, $P_{n,k} \geqslant N_{n,k}$.

(b) $D_{n,k}=0$ *precisely when* k *is odd and* n *is even.*

(c) $D_{n,2m}=\binom{[n/2]}{m}$, $D_{2l+1,2m+1}=\binom{l}{m}$.

Example. $D_{46,22}=1{,}352{,}078$, $D_{46,23}=0$. That is, if we expand a 46×46 determinant along the first 22 rows, we get 1 352 078 more positive terms than negative terms; if we expand along the first 23 rows, we get the same number of positive and negative terms.

Corollary 3.7. (a) *For any fixed* $m \geqslant 1$, $\lim_{n\to\infty} D_{n,2m}=\infty$, *and* $\lim_{l\to\infty} D_{2l+1,2m+1}=\infty$.

(b) *For any fixed* $k \geqslant 2$, $\lim_{n\to\infty} D_{n,k}/\binom{n}{k}=0$, *i.e., the difference between the numbers of the positive and negative terms is "small" in comparison with the total number of terms.*

Once $D_{n,k}$ are known, we can get closed formulae for $P_{n,k}$ and $N_{n,k}$ by Propositions 3.1(b) and 3.6.

Proposition 3.8. (a) $P_{n,2m}=\frac{1}{2}\left(\binom{n}{2m}+\binom{[n/2]}{m}\right)$, $N_{n,2m}=\frac{1}{2}\left(\binom{n}{2m}-\binom{[n/2]}{m}\right)$.

(b) $P_{2l,2m+1}=N_{2l,2m+1}=\frac{1}{2}\binom{2l}{2m+1}$.

(c) $P_{2l+1,2m+1}=\frac{1}{2}\left(\binom{2l+1}{2m+1}+\binom{l}{m}\right)$, $N_{2l+1,2m+1}=\frac{1}{2}\left(\binom{2l+1}{2m+1}-\binom{l}{m}\right)$.

Remark. As a direct corollary of Proposition 3.8(b), we get a "combinatorial proof" of that $\binom{2l}{2m+1}$, $\binom{2l+1}{2m+1}\pm\binom{l}{m}$ are all even, hence $\binom{2l+1}{2m+1}$ and $\binom{l}{m}$ have the same parity.

Now let us go back to the general case. When we expand an $n\times n$ determinant along arbitrary k rows, the following gives the general result, whose proof is easy and is omitted.

Proposition 3.9. *In* (1), *if*

$$\tau=\begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ i_1 & i_2 & \cdots & i_k & i'_1 & \cdots & i'_{n-k} \end{pmatrix}$$

is even (*or equivalently*, $(i_1-1)+(i_2-1)+\cdots+(i_k-k)$ *is even*), *then each term has the same sign as that of the corresponding one in* (2); *if* τ *is odd, then each term has the opposite sign.*

4. The generating functions

In this section we will deduce the generating functions and the "symmetrized" generating functions for $D_{n,k}$; hence, those for $P_{n,k}$ and $N_{n,k}$.

Proposition 4.1. *For fixed* $l\geqslant 0$,

(a) *The generating function* $D_{2l}(X)$ *for* $D_{2l,k}$, $k=0,1,\ldots,2l$, *is* $(1+X^2)^l$.

(b) *The generating function* $D_{2l+1}(X)$ *for* $D_{2l+1,k}$, $k=0,1,\ldots,2l+1$, *is* $(1+X)(1+X^2)^l$.

Proof. This follows from the expansions of $(1+X^2)^l$ and $(1+X)(1+X^2)^l$ and Fig. 3. $\square$

Remark. It is easy to check that we can rewrite the expressions for $D_{n,k}$ (in Proposition 3.6) in one form:

$$D_{n,k}=\frac{1+(-1)^{k(n-1)}}{2}\binom{[n/2]}{[k/2]}, \tag{3}$$

from which we can compute $D_n(X)$:

$$\begin{aligned} D_n(X)&=\sum_{m\geqslant 0}\left\{\binom{[n/2]}{m}X^{2m}+\frac{1+(-1)^{n+1}}{2}\binom{[n/2]}{m}X^{2m+1}\right\}\\ &=\left(1+\frac{1+(-1)^{n+1}}{2}X\right)\sum_{m\geqslant 0}\binom{[n/2]}{m}X^{2m}\\ &=\left(1+\frac{1+(-1)^{n+1}}{2}X\right)(1+X^2)^{[n/2]}. \end{aligned}$$

This combines (a) and (b) above.

Now let $D(X, Y)=\sum_{n\geq 0} D_n(X)Y^n=\sum_{n,k\geq 0} D_{n,k}X^kY^n$, the generating function for $D_{n,k}$. Then we have the following proposition.

Proposition 4.2.

$$D(X, Y)=\frac{1+Y+XY}{1-Y^2-X^2Y^2}.$$

Proof. By Proposition 4.1 above,

$$\begin{aligned} D_{2l}(X)Y^{2l}+D_{2l+1}(X)Y^{2l+1} &= (1+X^2)^lY^{2l}+(1+X^2)^l(1+X)Y^{2l+1} \\ &= (1+X^2)^lY^{2l}[1+(1+X)Y] \\ &= (Y^2+X^2Y^2)^l(1+Y+XY). \end{aligned}$$

$$\begin{aligned} D(X, Y) &= \sum_{l\geq 0} (D_{2l}(X)Y^{2l}+D_{2l+1}(X)Y^{2l+1}) \\ &= \sum_{l\geq 0} (Y^2+X^2Y^2)^l(1+Y+XY) \\ &= \frac{1+Y+XY}{1-Y^2-X^2Y^2}. \qquad \square \end{aligned}$$

Since the generating function $C_n(X)$ for $\binom{n}{k}$, $k=0,1,\ldots,n$, is easily seen to be $(1+X)^n$, and

$$C(X)=\sum_{n\geq 0} C_n(X)Y^n=\sum_{n,k\geq 0}\binom{n}{k}X^kY^n,$$

hence,

$$C(X)=\sum_{n\geq 0}(1+X)^nY^n=\frac{1}{1-Y-XY},$$

we get the corresponding generating functions $P_n(X)$, $P(X, Y)$ for $P_{n,k}$ and $N_n(X)$, $N(X, Y)$ for $N_{n,k}$ by Propositions 3.1, 4.1 and 4.2.

Proposition 4.3.

(a) $P_{2l}(X)=\dfrac{(1+X)^{2l}+(1+X^2)^l}{2}$,

$$P_{2l+1}(X)=(1+X)P_{2l}(X)=(1+X)\frac{(1+X)^{2l}+(1+X^2)^l}{2}.$$

(b) $N_{2l}(X)=\dfrac{(1+X)^{2l}-(1+X^2)^l}{2}$,

$$N_{2l+1}(X)=(1+X)N_{2l}(X)=(1+X)\frac{(1+X)^{2l}-(1+X^2)^l}{2},$$

(c) $P(X, Y)=\dfrac{1-Y^2-XY^2-X^2Y^2}{(1-Y-XY)(1-Y^2-X^2Y^2)}$,

(d) $N(X, Y)=\dfrac{XY^2}{(1-Y-XY)(1-Y^2-X^2Y^2)}$.

Proof. Straightforward. □

The following corollaries (to Propositions 4.1 and 4.3) show how the generating functions reveal the recurrence relations for $D_{n,k}$, $P_{n,k}$ and $N_{n,k}$.

Corollary 4.4. *For all n and k,*

$$D_{n,k}=D_{n-1,k}+(-1)^{k(n-1)}D_{n-1,k-1}.$$

Proof. By Proposition 4.1, $D_{2l+1}(X)=(1+X)D_{2l}(X)$, so for $n=2l+1$ odd, $D_{n,k}=D_{n-1,k}+D_{n-1,k-1}$. On the other hand, $D_{2l}(X)=(1+X^2)^l=(1+X^2)^{l-1}(1+X^2)=(1+X^2)^{l-1}(1+2X+X^2-2X)=(1+X)^2(1+X^2)^{l-1}-2X(1+X^2)^{l-1}=(1+X)D_{2l-1}(X)-2XD_{2l-2}(X)$, so for $n=2l$ even and $k=2m$ even, we still have $D_{n,k}=D_{n-1,k}+D_{n-1,k-1}$ since $XD_{2l-2}(X)$ is an odd function. For $n=2l$ even and $k=2m+1$ odd, $D_{2l,2m+1}=D_{2l-1,2m+1}+D_{2l-1,2m}-2D_{2l-2,2m}=D_{2l-1,2m+1}-D_{2l-1,2m}$, since $D_{2l-2,2m}=D_{2l-1,2m}$ by Proposition 3.4(A)$_m$. □

Remark. The entries where $-$ should be taken can be easily memorized: they are located simply at the bottom of the "diamonds" (in Fig. 4).

$$\begin{array}{ccc} & + & \\ + & & + \\ & - & \end{array}$$

Corollary 4.5. *For all n and k,*

(a) $$P_{n,k}=P_{n-1,k}+\frac{1+(-1)^{k(n-1)}}{2}P_{n-1,k-1}+\frac{1+(-1)^{k(n-1)-1}}{2}N_{n-1,k-1}.$$

(b) $$N_{n,k}=N_{n-1,k}+\frac{1+(-1)^{k(n-1)}}{2}N_{n-1,k-1}+\frac{1+(-1)^{k(n-1)-1}}{2}P_{n-1,k-1}.$$

Proof. (a) Since $P_{2l+1}(X)=(1+X)P_{2l}(X)$ (Proposition 4.3), $P_{n,k}=P_{n-1,k}+P_{n-1,k-1}$ for $n=2l+1$ odd. Since $P_{2l}(X)=(1+X)P_{2l-1}-X(1+X^2)^{l-1}$, we still have $P_{n,k}=P_{n-1,k}+P_{n-1,k-1}$ for $n=2l$ even and $k=2m$ even, by the oddness of

$X(1+X^2)^{l-1}$. For $n=2l$ even and $k=2m+1$ odd,

$$\begin{aligned}P_{2l,2m+1} &= P_{2l-1,2m+1}+P_{2l-1,2m}-\binom{l-1}{m}\\ &= P_{2l-1,2m+1}+P_{2l-1,2m}-D_{2l,2m}\\ &= P_{2l-1,2m+1}+N_{2l-1,2m}.\end{aligned}$$

(b) Similarly. □

Finally, the symmetric property of $P_{n,k}$, $N_{n,k}$ and $D_{n,k}$ (Proposition 3.5) suggests that if we parametrize the generating function by $n-k$ and k instead of n and k, we get symmetric functions in X and Y, i.e. function in the elementary symmetric functions $\sigma_1=X+Y$ and $\sigma_2=XY$.

Proposition 4.6. *Let* $C^*(X,Y)=\sum_{n,k\geq 0}\binom{n+k}{k}X^nY^k=\sum_{n,k\geq 0}\binom{n}{k}X^{n-k}Y^k$, $D^*(X,Y)=\sum_{n,k\geq 0}D_{n,k}X^{n-k}Y^k$, $P^*(X,Y)=\sum_{n,k\geq 0}P_{n,k}X^{n-k}Y^k$ *and* $N^*(X,Y)=\sum_{n,k\geq 0}N_{n,k}X^{n-k}Y^k$ *be the "symmetrized" generating function for* $\binom{n}{k}$, $D_{n,k}$, $P_{n,k}$ *and* $N_{n,k}$, *respectively. Then*

(a) $$C^*=\frac{1}{1-\sigma_2},$$

(b) $$D^*=\frac{1+\sigma_1}{1-\sigma_1^2+2\sigma_2},$$

(c) $$P^*=\frac{1-\sigma_1^2+\sigma_2}{(1-\sigma_1)(1-\sigma_1^2+2\sigma_2)},$$

(d) $$N^*=\frac{\sigma_2}{(1-\sigma_1)(1-\sigma_1^2+2\sigma_2)}.$$

Proof. Simply replace XY by X in Propositions 4.2 and 4.3. □

5. Applications

(1) The results in Section 3 can be applied to the expressions for the "shuffle products" [2, Section 3].

In a letter-place algebra $L[\mathscr{X}\,|\,\mathscr{U}]$,

$$\left(\begin{array}{cccccc|ccc}\dot{x}_{i_1} & \cdots & \dot{x}_{i_p} & x_{i_{p+1}} & \cdots & x_{i_s} & u_{j_1} & \cdots & u_{j_s}\\ \dot{x}_{l_1} & \cdots & \dot{x}_{l_q} & x_{l_{q+1}} & \cdots & x_{l_t} & u_{m_1} & \cdots & u_{m_t}\end{array}\right)$$
$$=\sum_{k=1}^{\binom{p+q}{p}}(-1)^{i(\sigma_k)}\left(\begin{array}{cccccc|ccc}x_{\sigma_k(i_1)} & \cdots & x_{\sigma_k(i_p)} & x_{i_{p+1}} & \cdots & x_{i_s} & u_{j_1} & \cdots & u_{j_s}\\ x_{\sigma_k(l_1)} & \cdots & x_{\sigma_k(l_q)} & x_{l_{q+1}} & \cdots & x_{l_t} & u_{m_1} & \cdots & u_{m_t}\end{array}\right),$$

where $\sigma_1,\ldots,\sigma_{\binom{p+q}{p}}$ are the distinguished representatives of $S_p\times S_q$ in S_{p+q}. Hence, the Straightening Formula [2, p. 68; 3, p. 13] can also be written in terms of representatives. The underlying idea is to *make* the expression *skew* relative to those "dotted" entries. However, as it is already skew relative to the entries in the same row (encoded in the bracket notation), we do not have to apply $S_p \times S_q$.

(2) Of course, this idea applies to other "symmetric/skew processes". For instance, to *permanents* [3, p. 21] and to *Pfaffians* [3, p. 59]. In the case of Pfaffians, the (parabolic) subgroup of S_{2r} is $S_2 \times S_2 \times \cdots \times S_2$ (r times), which has index $(2r)!/(2!)^r = r!(2r-1)!!$, divisible by $r!$.

(3) *Multiple Laplace pairings* [3, pp. 13–15] can be presented more rigorously using the distinguished representatives and the imbeddings

$$S_l \times S_{k-l} \times S_{n-k} \subsetneq S_k \times S_{n-k} \subsetneq S_n.$$

(4) If we substitute the expressions in Propositions 3.6 (c) and 3.8 into Proposition 3.3′ or substitute formula (3) (a unified expression for $D_{n,k}$) and the counterparts for $P_{n,k}$ and $N_{n,k}$ into Proposition 3.3′, we get a few more combinatorial identities.

(5) In [6, Ch. 3], all k-subsets of an n-set is linearly ordered in such a way that each subset is obtained from its immediate predecessor by deleting a single element and adjoining another element. Now we can ask the following *question*: Can we do it in such a way that each adjoined element is either an immediate predecessor or an immediate successor (i.e., a *neighbor*) of the deleted one relative a prefixed linear order on the n-set?

The answer is yes to the trivial cases: $k = 0, 1, n-1, n$. However, we have the following proposition.

Proposition 5.1. *When* $D_{n,k} > 1$, *this is* impossible.

Proof. There is a natural bijection between the set of the distinguished representatives and the set of k-subset, viz.

$$\{i_1, \ldots, i_k\} \mapsto \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ i_1 & i_2 & \cdots & i_k & i'_1 & \cdots & i'_{n-k} \end{pmatrix},$$

where $i_1 < \cdots < i_k$, $i'_1 < \cdots < i'_k$, with inverse $\sigma \mapsto \{\sigma(1), \ldots, \sigma(k)\}$. So each subset is assigned a sign (via this bijection). Each exchange as described involves a sign change, and if $D_{n,k} > 1$, there is no way to alter the signs $\binom{n}{k} - 1$ times. □

Note that in the trivial cases $k = 0, 1, n-1, n, D_{n,k}$ all equals 0 or 1. So the only remaining cases are n even and k odd with $3 \leqslant k \leqslant n-3$.

Acknowledgment

We would like to thank Herbert Wilf for giving an interesting talk at West Chester University that initiated this work, Shiv Gupta for many helpful conversations and Hal Switkay for showing us that we have the "diamonds" in Fig. 4.

References

[1] N. Bourbaki, Groupes et Algèbra de Lie, Chaps. IV–VI (Hermann, Paris, 1968).
[2] J. Désarménien, J. Kung and G.-C. Rota, Invariant theory, Young tableaux, and combinatorics, Adv. Math. 27 (1978) 63–92.
[3] F. Grosshans, G.-C. Rota and J. Stein, Invariant Theory and Superalgebras (AMS, Providence, RI, 1987).
[4] F. Hohn, Elementary Matrix Algebra (Macmillan, London, 3rd Ed., 1973).
[5] N. Jacobson, Basic Algebra I (Freeman, San Francisco, CA, 2nd Ed., 1985).
[6] A. Nijenhuis and H. Wilf, Combinatorial Algorithm (Academic Press, New York, 2nd Ed., 1978).